Archive of Applied Mechanics (2022) 92:679–689
https://doi.org/10.1007/s00419-022-02105-5

RAPID COMMUNICATION

Rubens G. Salsa Jr. · Daniel T. Kawano · Fai Ma

Rayleigh's quotients and eigenvalue bounds for linear dynamical systems

Received: 16 November 2021 / Accepted: 4 January 2022 / Published online: 17 January 2022

Abstract The primary objective of this article is to demonstrate that Rayleigh's quotient and its variants retain the usual properties of boundedness and stationarity even when the linear vibratory system is non-classically damped, extending previously accepted results that these quotients could attain stationarity when damping was proportional or the modal damping matrix was diagonally dominant. This conclusion is reached by allowing the quotients to be defined in complex space and using complex differentiation. A secondary objective is to show how these quotients and their associated eigenvalue problems can be combined to generate bounds on the system's eigenvalues, an immediate consequence that follows from establishing boundedness and stationarity in complex space. The reported bounds are simple to compute and appear to be tighter than previous bounds reported in the literature.

Keywords Linear dynamical systems · Damping · Rayleigh's quotient · Rayleigh's principle · Eigenvalue bounds

1 Introduction

The equation of motion of linear systems is one of the most commonly used equations in science and engineering. Of particular significance in the study of linear vibrations and structural dynamics is the class of passive systems characterized by three symmetric and positive-definite matrices, i.e., the mass, damping, and stiffness matrices. For brevity, this class of systems will be referred to as damped linear systems.

It is customary to analyze a linear system with the proportion between potential and kinetic energies; a ratio that involves the stiffness and mass matrices is termed as Rayleigh's quotient. In real space, this function has been shown to be bounded by the lowest and highest natural frequencies, and it attains a stationary value at each real mode shape. In a sufficiently small neighborhood of a stationary point, Rayleigh's quotient does not vary significantly and there is loss of sensitivity. This set of properties drives the application of Rayleigh's quotient in various fields of science and engineering. For example, Rayleigh's quotient plays a central role in

R. G. Salsa Jr. (✉)
Department of Mechanical Engineering, Federal University of Rio Grande do Norte, Natal, Rio Grande do Norte 59078-970, Brazil
E-mail: rubens.salsa@ufrn.br

D. T. Kawano
Department of Mechanical Engineering, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, USA
E-mail: kawano@rose-hulman.edu

F. Ma
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA
E-mail: fma@berkeley.edu

the characterization of natural frequencies of linear systems [1,2]. In addition, it is the foundation of iterative processes to determine numerical solutions for classes of eigenvalue problems [3,4].

In the presence of viscous damping, Phani and Adhikari [5] showed that a dissipative system is also associated with two other Rayleigh-like quotients, each involving the damping matrix and either the stiffness or mass matrix as well. For simplicity, all three variants of quotients will be referred to as Rayleigh's quotients. In their article, the authors showed that these quotients attain stationarity if the system is proportionally damped. In the case of a non-proportionally damped system, they were able to show that the two quotients involving the damping matrix have stationary points if the modal damping matrix is diagonally dominant. Because, in general, a damped linear system does not possess real normal modes, i.e., the system's eigenvectors do not coincide with the normal modes, their derivation was based on a first-order approximation of the complex modes by the undamped modes [6]. Since the system's eigenvectors are often complex, any further investigation of the properties of Rayleigh's quotients would require full consideration of complex input vectors. In view of this, the primary objective of this article is to completely establish the properties of stationarity and boundedness of Rayleigh's quotients for damped linear systems using complex algebra. The use of complex differentiation allows the analysis to bypass the usual assumptions of proportional damping or diagonal dominance.

The secondary objective is to demonstrate that these quotients lead rather intuitively to geometrical bounds in the complex plane for the system eigenvalues, which is an immediate consequence of establishing the boundedness property of the Rayleigh's quotients in complex space. Estimates on the location of eigenvalues are important in design, modeling, and simulation, especially for large structural systems. For example, the range of the real parts of the system eigenvalues is of important consideration in numerical simulation because they offer an approximation of the stiffness of the differential equations. Similarly, the spread of the imaginary parts represents the frequency interval for which an external excitation may induce resonance [7]. Eigenvalue bounds may also be used in the design of active vibration controllers [8] and to predict response bounds [9,10].

Other researchers have reported on eigenvalue bounds for linear systems [7,8,11–13]. The Gershgorin theorem has been used [11,13] in order to compute a circle in the complex plane enclosing the system eigenvalues. Lancaster [11] also used perturbation theory to compute tighter bounds for the real and imaginary part of system eigenvalues. Nicholson and Inman [12] used the Cauchy–Schwarz inequality to derive a rectangle in the complex plane containing the system eigenvalues. Later, Nicholson [7] obtained sharper bounds using a modified state vector dependent on a parameter. However, bounds derived herein appear to be tighter than these previously reported bounds and, most importantly, are appreciably more simple to compute.

The organization of this paper is as follows. A concise review of conservative (i.e., undamped) systems and the classical Rayleigh's quotient is presented in Sect. 2 to establish mathematical preliminaries for subsequent analysis. Stationarity and boundedness of Rayleigh's quotients for dissipative systems are demonstrated in Sect. 3. It is then shown in Sect. 4 how these quotients can be used to obtain both established and novel bounds for the eigenvalues of a damped linear system. An illustrative example is provided in Sect. 5 to demonstrate the findings herein and compare them to prior results. The paper closes with a summary of the main findings in Sect. 6.

2 Conservative systems

To provide the necessary mathematical background before discussing Rayleigh-like quotients for damped linear systems, a brief review of undamped systems and the classical Rayleigh's quotient is given here.

2.1 The equation of motion and its spectral properties

The equation of motion of an n-degree-of-freedom conservative linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are $n \times n$ real, symmetric, and positive-definite matrices; they are referred to as the mass and stiffness matrices, respectively. The n-dimensional vector $\mathbf{q}$ contains the n generalized coordinates describing the behavior of system (1).

Associated with Eq. (1) is the generalized eigenvalue problem

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{0} \tag{2}$$

with eigenvalue ω^2 and n-dimensional eigenvector $\mathbf{u}$. Owing to symmetry and positive definiteness of $\mathbf{M}$ and $\mathbf{K}$, all eigenvalues ω_j^2 $(j = 1, 2, \ldots, n)$ are real and positive, and the corresponding eigenvectors $\mathbf{u}_j$ are real and linearly independent [4]. The eigenvectors $\mathbf{u}_j$ are mode shapes and ω_j^2 are the square of natural frequencies of vibration. By convention, the natural frequencies ω_j are arranged in order of increasing magnitude: $0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_n$. The mode shapes are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$, and they can be normalized, for convenience, with respect to $\mathbf{M}$ so that

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_k = \delta_{jk}, \quad \mathbf{u}_j^{\mathrm{T}}\mathbf{K}\mathbf{u}_k = \delta_{jk}\omega_j^2, \tag{3}$$

where $\delta_{jk} = 0$ for $j \neq k$ and $\delta_{jk} = 1$ for $j = k$. In such case, the mode shapes are said to be mass normalized. It should be noted that the conditions in Eq. (3) hold whether or not there are repeated eigenvalues [14].

2.2 The classical Rayleigh's quotient

Rayleigh's quotient is commonly defined as the ratio between the potential and kinetic energies of system (1):

$$R(\mathbf{x}) = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{K}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}}, \tag{4}$$

where $\mathbf{x}$ is an n-dimensional real vector. This function is well-defined for any $\mathbf{x} \neq \mathbf{0}$ due to positive definiteness of $\mathbf{M}$. It is clear from Eqs. (3) and (4) that if the quotient is evaluated at a mode shape $\mathbf{u}_j$, then

$$R(\mathbf{u}_j) = \omega_j^2. \tag{5}$$

The classical Rayleigh's principle states that in real space, Rayleigh's quotient (4) attains a stationary value at each mode shape and is bounded by the square of the lowest and highest natural frequencies of system (1). The proofs for stationarity and boundedness of Rayleigh's quotient are standard in engineering literature, but they are repeated here because the subsequent derivation for quotients associated with dissipative systems will follow a similar strategy.

2.2.1 Stationarity

Starting with stationarity, define the derivative of a multivariate scalar function F with respect to a real n-dimensional vector $\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]^{\mathrm{T}}$ as

$$\frac{\partial F}{\partial \mathbf{x}} = \left[\frac{\partial F}{\partial x_1} \; \frac{\partial F}{\partial x_2} \cdots \frac{\partial F}{\partial x_n}\right]^{\mathrm{T}}. \tag{6}$$

Rayleigh's quotient has a stationary value at $\mathbf{x}$ if

$$\frac{\partial R}{\partial \mathbf{x}} = \mathbf{0}. \tag{7}$$

Since

$$\frac{\partial R}{\partial \mathbf{x}} = \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}} \left(\mathbf{K} - R(\mathbf{x})\mathbf{M}\right)\mathbf{x}, \tag{8}$$

Eqs. (2), (3), (5), and (8) imply

$$\frac{\partial}{\partial \mathbf{x}} R(\mathbf{u}_j) = \frac{2}{\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j} \left(\mathbf{K} - \omega_j^2\mathbf{M}\right)\mathbf{u}_j = \mathbf{0}. \tag{9}$$

Therefore, $R(\mathbf{x})$ attains a stationary value at each mode shape $\mathbf{u}_j$. A stationary point may be a minimum, a maximum, or a saddle point.

2.2.2 Boundedness

Next, to demonstrate boundedness, one can start by using modal expansion to express any nonzero $\mathbf{x}$ as

$$\mathbf{x} = \sum_{j=1}^{n} c_j \mathbf{u}_j, \tag{10}$$

where c_j are real scalar coefficients. This implies, from Eqs. (4) and (5) that

$$R(\mathbf{x}) = \frac{\sum_{j=1}^{n} c_j^2 \omega_j^2}{\sum_{j=1}^{n} c_j^2}. \tag{11}$$

Rewrite Eq. (11) as [1]

$$R(\mathbf{x}) = \omega_1^2 \frac{\sum_{j=1}^{n} c_j^2 \left(\frac{\omega_j}{\omega_1}\right)^2}{\sum_{j=1}^{n} c_j^2}. \tag{12}$$

The numerator of Eq. (12) is greater than the denominator, which means $R(\mathbf{x}) \geq \omega_1^2$, and thus Rayleigh's quotient attains, in real space, a global minimum at the first mode shape $\mathbf{u}_1$. Similarly, Eq. (11) can be written as

$$R(\mathbf{x}) = \omega_n^2 \frac{\sum_{j=1}^{n} c_j^2 \left(\frac{\omega_j}{\omega_n}\right)^2}{\sum_{j=1}^{n} c_j^2}, \tag{13}$$

in which case the numerator is smaller than the denominator, and so $R(\mathbf{x}) \leq \omega_n^2$. Therefore, Rayleigh's quotient is bounded by the square of the lowest and highest natural frequencies:

$$\omega_1^2 \leq R(\mathbf{x}) \leq \omega_n^2. \tag{14}$$

3 Dissipative systems

Damped linear systems and their associated Rayleigh-like quotients are next investigated.

3.1 The equation of motion and its spectral properties

An n-degree-of-freedom damped linear system has the equation of motion

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{15}$$

where $\mathbf{C}$ is the $n \times n$ real, symmetric, and positive-definite damping matrix. In this case, Eq. (15) is associated with the quadratic eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0} \tag{16}$$

with eigenvalue λ and n-dimensional eigenvector $\mathbf{v}$, which are generally complex. Solution of Eq. (16) yields $2n$ eigenvalues λ_j $(j = 1, 2, \ldots, 2n)$ and their corresponding eigenvectors $\mathbf{v}_j$, which can only be determined up to an arbitrary scalar multiplier. The spectrum of a damped linear system (15) is composed of negative real eigenvalues, associated with exponential decay of the response, and complex conjugate eigenvalues, whose negative real part is a rate of decay and whose imaginary part is a damped frequency of vibration [15]. The real eigenvalues can always have real eigenvectors associated with them.

3.2 Rayleigh's quotients

The dissipative system (15) is associated with three Rayleigh-like quotients [5]

$$R_{\mathbf{AB}}(\mathbf{z}) = \frac{\mathbf{z}^*\mathbf{Az}}{\mathbf{z}^*\mathbf{Bz}} \tag{17}$$

formed by replacing the $n \times n$ real, symmetric, and positive-definite matrices $\mathbf{A}$ and $\mathbf{B}$ with some selection of the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. For simplicity, quotients of the form in Eq. (17) will be referred to as Rayleigh's quotients. Because the eigenvectors of Eq. (16) are generally complex, the quotients in Eq. (17) are functions of n-dimensional complex vectors $\mathbf{z} \neq \mathbf{0}$, where $\mathbf{z}^* = \bar{\mathbf{z}}^{\mathrm{T}}$ denotes the complex conjugate transpose. While $\mathbf{z}$ may be complex, the quotient $R_{\mathbf{AB}}$ is real.

Associated with each Rayleigh's quotient (17) is the generalized eigenvalue problem

$$(\mathbf{A} + \mu^{\mathbf{AB}}\mathbf{B})\mathbf{w}^{\mathbf{AB}} = \mathbf{0} \tag{18}$$

with eigenvalue $\mu^{\mathbf{AB}}$ and n-dimensional eigenvector $\mathbf{w}^{\mathbf{AB}}$. Similar to the eigenvalue problem (2), because both $\mathbf{A}$ and $\mathbf{B}$ are symmetric and positive definite, $-\mu_j^{\mathbf{AB}}$ $(j = 1, 2, \ldots, n)$ are all real and positive. Thus, all eigenvalues $\mu_j^{\mathbf{AB}}$ are negative real numbers, and they can be arranged for convenience in order of increasing magnitude: $0 \leq |\mu_1^{\mathbf{AB}}| \leq |\mu_2^{\mathbf{AB}}| \leq \ldots \leq |\mu_n^{\mathbf{AB}}|$. Moreover, the corresponding eigenvectors $\mathbf{w}_j^{\mathbf{AB}}$ are real, linearly independent, and orthogonal with respect to both $\mathbf{A}$ and $\mathbf{B}$:

$$(\mathbf{w}_j^{\mathbf{AB}})^{\mathrm{T}}\mathbf{B}\mathbf{w}_k^{\mathbf{AB}} = \delta_{jk}, \quad (\mathbf{w}_j^{\mathbf{AB}})^{\mathrm{T}}\mathbf{A}\mathbf{w}_k^{\mathbf{AB}} = \delta_{jk}|\mu_j^{\mathbf{AB}}|, \tag{19}$$

where $\mathbf{w}_j^{\mathbf{AB}}$ have been normalized with respect to $\mathbf{B}$ for convenience. Consequently, by examining Eq. (18) for the jth eigensolution, pre-multiplying by $(\mathbf{w}_j^{\mathbf{AB}})^* = (\mathbf{w}_j^{\mathbf{AB}})^{\mathrm{T}}$, and rearranging, it can be shown that

$$\frac{(\mathbf{w}_j^{\mathbf{AB}})^{\mathrm{T}}\mathbf{A}\mathbf{w}_j^{\mathbf{AB}}}{(\mathbf{w}_j^{\mathbf{AB}})^{\mathrm{T}}\mathbf{B}\mathbf{w}_j^{\mathbf{AB}}} = R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}}) = -\mu_j^{\mathbf{AB}} = |\mu_j^{\mathbf{AB}}|. \tag{20}$$

The properties of stationarity and boundedness of the Rayleigh's quotients (17) will now be derived. It should be emphasized that the only restrictions on the real $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are that they are symmetric and positive definite.

3.2.1 Stationarity

To prove stationarity of $R_{\mathbf{AB}}(\mathbf{z})$, complex differentiation is needed. Firstly, note that any complex vector $\mathbf{z}$ can be expressed in rectangular form as $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are n-dimensional real vectors, and so $R_{\mathbf{AB}}(\mathbf{z}) = R_{\mathbf{AB}}(\mathbf{x}, \mathbf{y})$. The multivariate real-valued function $R_{\mathbf{AB}}(\mathbf{x}, \mathbf{y})$ attains a stationary value when [16]

$$\frac{\partial R}{\partial \mathbf{x}} = \frac{\partial R}{\partial \mathbf{y}} = \mathbf{0}. \tag{21}$$

Because $\mathbf{z}^*\mathbf{Pz} = \mathbf{x}^{\mathrm{T}}\mathbf{Px} + \mathbf{y}^{\mathrm{T}}\mathbf{Py}$ for a real symmetric matrix $\mathbf{P}$, it follows that

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}}(\mathbf{z}^*\mathbf{Pz}) &= \frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^{\mathrm{T}}\mathbf{Px}) = 2\mathbf{Px}, \\ \frac{\partial}{\partial \mathbf{y}}(\mathbf{z}^*\mathbf{Pz}) &= \frac{\partial}{\partial \mathbf{y}}(\mathbf{y}^{\mathrm{T}}\mathbf{Py}) = 2\mathbf{Py}. \end{aligned} \tag{22}$$

Consequently,

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} R_{\mathbf{AB}}(\mathbf{z}) &= \frac{2}{\mathbf{z}^*\mathbf{Bz}} \left(\mathbf{A} - R_{\mathbf{AB}}(\mathbf{z})\mathbf{B}\right)\mathbf{x}, \\ \frac{\partial}{\partial \mathbf{y}} R_{\mathbf{AB}}(\mathbf{z}) &= \frac{2}{\mathbf{z}^*\mathbf{Bz}} \left(\mathbf{A} - R_{\mathbf{AB}}(\mathbf{z})\mathbf{B}\right)\mathbf{y}. \end{aligned} \tag{23}$$

Finally, if $\mathbf{x} = \mathbf{w}_j^{\mathbf{AB}}$ and $\mathbf{y} = \mathbf{0}$ so that $\mathbf{z} = \mathbf{w}_j^{\mathbf{AB}}$, then

$$\frac{\partial}{\partial \mathbf{x}} R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}} + i\mathbf{0}) = \frac{\partial}{\partial \mathbf{y}} R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}} + i\mathbf{0}) = \mathbf{0}. \tag{24}$$

Therefore, $\mathbf{z} = \mathbf{w}_j^{\mathbf{AB}}$ is a critical point of $R_{\mathbf{AB}}(\mathbf{z})$. In contrast with the work of Phani and Adhikari [5], this result does not require an assumption that the modal damping matrix (i.e., the damping matrix transformed by the real normal modes $\mathbf{u}_j$) be diagonally dominant.

Note that other critical points such as $i\mathbf{w}_j^{\mathbf{AB}}$ and $\mathbf{w}_j^{\mathbf{AB}} + i\mathbf{w}_j^{\mathbf{AB}} = (1+i)\mathbf{w}_j^{\mathbf{AB}}$ lie in the subspace generated by the real eigenvector $\mathbf{w}_j^{\mathbf{AB}}$. In particular, consider complex vectors of the form $\mathbf{z} = \mathbf{x} + i\mathbf{y} = \mathbf{w}_j^{\mathbf{AB}} e^{i\theta_j}$ so that $\mathbf{x} = \mathbf{w}_j^{\mathbf{AB}} \cos\theta_j$ and $\mathbf{y} = \mathbf{w}_j^{\mathbf{AB}} \sin\theta_j$. It can be verified that $R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}} e^{i\theta_j}) = |\mu_j^{\mathbf{AB}}|$, which means

$$\frac{\partial}{\partial \mathbf{x}} R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}} e^{i\theta_j}) = \frac{\partial}{\partial \mathbf{y}} R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}} e^{i\theta_j}) = \mathbf{0}. \tag{25}$$

Consequently, the circle $\mathbf{z} = \mathbf{w}_j^{\mathbf{AB}} e^{i\theta_j}$ in complex space is a continuous locus of critical points, along which $R_{\mathbf{AB}}(\mathbf{w}_j^{\mathbf{AB}} e^{i\theta_j}) = |\mu_j^{\mathbf{AB}}|$ is constant. These circles are level curves of $R_{\mathbf{AB}}(\mathbf{z})$. When $\mathbf{A} = \mathbf{K}$ and $\mathbf{B} = \mathbf{M}$, the quotient $R_{\mathbf{KM}}(\mathbf{z})$ is equivalent to the traditional Rayleigh's quotient (4) but defined on a complex domain, and thus the classical result that Rayleigh's quotient attains a global minimum at the first normal mode $\mathbf{z} = \mathbf{u}_1$ is correct in real space only.

3.2.2 Boundedness

To demonstrate boundedness of $R_{\mathbf{AB}}(\mathbf{z})$ in complex space, begin by expressing the complex vector $\mathbf{z}$ as a linear combination of the real eigenvectors $\mathbf{w}_j^{\mathbf{AB}}$:

$$\mathbf{z} = \sum_{j=1}^{n} c_j \mathbf{w}_j^{\mathbf{AB}}, \tag{26}$$

where the coefficients c_j are complex. As a result, from Eqs. (17) and (19),

$$R_{\mathbf{AB}}(\mathbf{z}) = \frac{\sum\limits_{j=1}^{n} |c_j|^2 |\mu_j^{\mathbf{AB}}|}{\sum\limits_{j=1}^{n} |c_j|^2}, \tag{27}$$

and so, by the same processes outlined in Sect. 2.2.2, the Rayleigh's quotients (17) are bounded by the smallest and largest eigenvalues, in magnitude, of their associated eigenvalue problem (18):

$$|\mu_1^{\mathbf{AB}}| \leq R_{\mathbf{AB}}(\mathbf{z}) \leq |\mu_n^{\mathbf{AB}}|. \tag{28}$$

4 Rayleigh's quotients and eigenvalue bounds

Interestingly, the boundedness property of the Rayleigh's quotients (17) can be used to obtain simple bounds for the eigenvalues of system (15) rather intuitively. Bounds for complex eigenvalues are considered first, followed by an investigation into bounds for real eigenvalues.

4.1 Bounds for complex eigenvalues

Suppose λ_j is a complex eigenvalue of the quadratic eigenvalue problem (16). Pre-multiplying Eq. (16) by the conjugate transpose $\mathbf{v}_j^*$ yields

$$\mathbf{v}_j^*\mathbf{M}\mathbf{v}\lambda_j^2 + \mathbf{v}_j^*\mathbf{C}\mathbf{v}\lambda_j + \mathbf{v}_j^*\mathbf{K}\mathbf{v} = 0. \tag{29}$$

If the conjugate transpose of Eq. (29) is taken, then one obtains

$$\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j\bar{\lambda}_j^2 + \mathbf{v}_j^*\mathbf{C}\mathbf{v}_j\bar{\lambda}_j + \mathbf{v}_j^*\mathbf{K}\mathbf{v}_j = 0 \tag{30}$$

because the coefficient matrices are real and symmetric. Therefore, the complex conjugate pair $(\lambda_j, \bar{\lambda}_j)$ are the roots of

$$\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j\lambda^2 + \mathbf{v}_j^*\mathbf{C}\mathbf{v}_j\lambda + \mathbf{v}_j^*\mathbf{K}\mathbf{v}_j = 0, \tag{31}$$

and so

$$\lambda_j,\ \bar{\lambda}_j = \frac{-\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j \pm \sqrt{(\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j)^2 - 4(\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j)(\mathbf{v}_j^*\mathbf{K}\mathbf{v}_j)}}{2\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j}, \tag{32}$$

where $(\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j)^2 - 4(\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j)(\mathbf{v}_j^*\mathbf{K}\mathbf{v}_j) < 0$. Alternatively, in terms of the Rayleigh's quotients (17),

$$\lambda_j,\ \bar{\lambda}_j = -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2} \pm i\sqrt{R_{\mathbf{KM}}(\mathbf{v}_j)}\sqrt{1 - \frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{4R_{\mathbf{KC}}(\mathbf{v}_j)}}. \tag{33}$$

4.1.1 An annular region in the complex plane

From Eq. (32), it can be shown that the product of a complex conjugate pair of eigenvalues is directly related to the quotient $R_{\mathbf{KM}}(\mathbf{z})$:

$$\lambda_j\bar{\lambda}_j = |\lambda_j|^2 = \frac{\mathbf{v}_j^*\mathbf{K}\mathbf{v}_j}{\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j} = R_{\mathbf{KM}}(\mathbf{v}_j) = R_{\mathbf{KM}}(\bar{\mathbf{v}}_j). \tag{34}$$

Comparing the eigenvalue problems (2) and (18), it is clear that $\mu_j^{\mathbf{KM}} = -\omega_j^2$, and so it follows from Eq. (28) that

$$\omega_1 \le |\lambda_j| \le \omega_n. \tag{35}$$

In words, Eq. (35) defines an annular region in the complex plane in which all complex eigenvalues must reside, where the inner and outer radii are given by the lowest and highest natural frequencies of system (15). This result is surprising and remarkable because the quotient $R_{\mathbf{KM}}(\mathbf{z})$ is defined without recourse to the damping matrix $\mathbf{C}$, but when evaluated at a complex eigenvector $\mathbf{v}_j$ of the quadratic eigenvalue problem (16), $R_{\mathbf{KM}}(\mathbf{v}_j)$ is connected with the associated complex eigenvalue λ_j.

On a final note, the result in Eq. (35) is not new. The bounds in Eq. (35) were previously obtained by Lancaster using perturbation theory [11], but the novel methodology presented here is more intuitive and elementary because it follows directly from the well-established concepts presented in Sects. 2 and 3.

4.1.2 A rectangular region in the complex plane

Upon further examination of Eq. (33), it is possible to obtain another set of simple bounds on the complex eigenvalues. Starting with the real part of the eigenvalues, from Eq. (33),

$$\mathrm{Re}(\lambda_j) = -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2} < 0. \tag{36}$$

Thus, Eq. (28) implies

$$\frac{\mu_n^{\mathbf{CM}}}{2} \le \mathrm{Re}(\lambda_j) \le \frac{\mu_1^{\mathbf{CM}}}{2}. \tag{37}$$

As with Eq. (35), this result is not new and appears in Nicholson's work [7], but the simplicity by which it follows from the boundedness property of the Rayleigh's quotients (17) is noteworthy.

A previously unreported bound for the imaginary parts of the complex eigenvalues is now presented. First, define the ratio

$$x = \frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{4R_{\mathbf{KC}}(\mathbf{v}_j)} \tag{38}$$

so that the imaginary part of Eq. (33), in magnitude, can be written as

$$\left|\mathrm{Im}(\lambda_j)\right| = \sqrt{R_{\mathbf{KM}}(\mathbf{v}_j)}\sqrt{1-x}. \tag{39}$$

Note that $0 < x < 1$ because $\sqrt{1-x}$ must be real, and so $\sqrt{1-x}$ may be expanded as a binomial series [17]:

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \cdots. \tag{40}$$

The upper bound on $\left|\mathrm{Im}(\lambda_j)\right|$ in Eq. (39) is approached when $x \to 0$. Thus, considering very small x (i.e., $0 < x \ll 1$), the series for $\sqrt{1-x}$ in Eq. (40) can be approximated well by neglecting all terms with powers of x greater than 1, resulting in

$$\sqrt{1-x} < 1 - \frac{x}{2}. \tag{41}$$

Consequently,

$$0 < \left|\mathrm{Im}(\lambda_j)\right| < \sqrt{R_{\mathbf{KM}}(\mathbf{v}_j)}\left(1 - \frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{8R_{\mathbf{KC}}(\mathbf{v}_j)}\right), \tag{42}$$

and hence, by Eq. (28),

$$0 < \left|\mathrm{Im}(\lambda_j)\right| < \omega_n\left(1 - \frac{\mu_1^{\mathbf{CM}}}{8\mu_n^{\mathbf{KC}}}\right). \tag{43}$$

Numerical simulations suggest that these bounds on the imaginary parts of the complex eigenvalues are tighter than those reported by Nicholson [7]. This may in part be due to the use of the extreme eigenvalue $\mu_n^{\mathbf{KC}}$ of the eigenvalue problem (18) involving $\mathbf{A} = \mathbf{K}$ and $\mathbf{B} = \mathbf{C}$, which Nicholson does not consider.

When taken together, Eqs. (37) and (39) define a rectangular region in the complex plane containing the complex eigenvalues. The intersection of this rectangular region and the annular region discussed in Sect. 4.1.1 yields a more refined result at little additional computational cost; calculation of extreme eigenvalues for large systems may be efficiently achieved using computational packages that can return only the smallest and largest eigenvalues, such as `eigs` in MATLAB.

4.2 Bounds for real eigenvalues

The Rayleigh's quotients (17) will now be used to develop previously unreported bounds for the real eigenvalues of system (15). Recall that complex eigenvalues of the quadratic eigenvalue problem (16) occur in conjugate pairs and thus they are both roots of Eq. (31). This is not the case for real eigenvalues. If λ_j is a real eigenvalue of Eq. (16) with real eigenvector $\mathbf{v}_j$, then, analogous to Eq. (31), it is one of the roots of

$$\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j\lambda^2 + \mathbf{v}_j^{\mathrm{T}}\mathbf{C}\mathbf{v}_j\lambda + \mathbf{v}_j^{\mathrm{T}}\mathbf{K}\mathbf{v}_j = 0. \tag{44}$$

The other root of Eq. (44), say ρ_j, is generally not an eigenvalue unless λ_j is a repeated real eigenvalue. From Eq. (44), both λ_j and ρ_j can be expressed as

$$\lambda_j,\ \rho_j = \frac{-\mathbf{v}_j^{\mathrm{T}}\mathbf{C}\mathbf{v}_j \pm \sqrt{(\mathbf{v}_j^{\mathrm{T}}\mathbf{C}\mathbf{v}_j)^2 - 4(\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j)(\mathbf{v}_j^{\mathrm{T}}\mathbf{K}\mathbf{v}_j)}}{2\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j}, \tag{45}$$

where $(\mathbf{v}_j^{\mathrm{T}}\mathbf{C}\mathbf{v}_j)^2 - 4(\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j)(\mathbf{v}_j^{\mathrm{T}}\mathbf{K}\mathbf{v}_j) > 0$. Rewriting Eq. (45) in terms of the Rayleigh's quotients (17),

$$\lambda_j,\ \rho_j = -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2}\left(1 \mp \sqrt{1 - \frac{4R_{\mathbf{KC}}(\mathbf{v}_j)}{R_{\mathbf{CM}}(\mathbf{v}_j)}}\right). \tag{46}$$

Similar to the process in Sect. 4.1.2, define the ratio

$$y = \frac{4R_{\mathbf{KC}}(\mathbf{v}_j)}{R_{\mathbf{CM}}(\mathbf{v}_j)}, \tag{47}$$

where $0 < y < 1$ because the real eigenvalues λ_j are strictly negative, and so Eq. (46) becomes

$$\lambda_j,\ \rho_j = -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2}\left(1 \mp \sqrt{1-y}\right). \tag{48}$$

The extremes of Eq. (48), and thus the upper and lower bounds for λ_j, are approached as $y \to 0$, and so Eq. (48) will be examined with very small y (i.e., $0 < y \ll 1$). Consequently, as in Eq. (41), $\sqrt{1-y}$ will be approximated at the extremes of Eq. (48) by a truncated binomial series:

$$\sqrt{1-y} < 1 - \frac{y}{2}. \tag{49}$$

Equations (47)–(49) and applying Eq. (28) reveal that an upper bound for λ_j is such that

$$\begin{aligned} \lambda_j &< -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2}\left(1 - \sqrt{1-y}\right) \\ &< -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2}\left(\frac{y}{2}\right) \\ &< R_{\mathbf{KC}}(\mathbf{v}_j) \\ &< \mu_1^{\mathbf{KC}}. \end{aligned} \tag{50}$$

Likewise, for a lower bound on λ_j,

$$\begin{aligned} \lambda_j &> -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2}\left(1 + \sqrt{1-y}\right) \\ &> -\frac{R_{\mathbf{CM}}(\mathbf{v}_j)}{2}\left(2 - \frac{y}{2}\right) \\ &> R_{\mathbf{CM}}(\mathbf{v}_j) - R_{\mathbf{KC}}(\mathbf{v}_j) \\ &> \mu_n^{\mathbf{CM}} - \mu_1^{\mathbf{KC}}. \end{aligned} \tag{51}$$

Therefore, by combining Eqs. (50) and (51), every real eigenvalue λ_j of Eq. (16) is bounded as follows:

$$\mu_n^{\mathbf{CM}} - \mu_1^{\mathbf{KC}} < \lambda_j < \mu_1^{\mathbf{KC}}. \tag{52}$$

As with Eq. (43), the novel bounds in Eq. (52) for the real eigenvalues of system (15) seem to be tighter than those given by Nicholson [7] based on numerical simulations. Again, these bounds feature an extreme eigenvalue (in this case, $\mu_1^{\mathbf{KC}}$) not present in Nicholson's work.

5 An illustrative example

Consider a system of the form (15) with coefficients $\mathbf{M} = \mathbf{I}$ and

$$\mathbf{C} = \begin{bmatrix} 4 & 1 & -2 & 0 \\ 1 & 4 & 1 & -1 \\ -2 & 1 & 6 & 0 \\ 0 & -1 & 0 & 4 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 10 & 6 & -3 & -5 \\ 6 & 10 & -2 & -5 \\ -3 & -2 & 6 & 0 \\ -5 & -5 & 0 & 8 \end{bmatrix}.$$

This system is neither proportionally damped nor, more generally, classically damped (i.e., $\mathbf{CM}^{-1}\mathbf{K} \neq \mathbf{KM}^{-1}\mathbf{C}$ [18]).

Solving the associated quadratic eigenvalue problem (16) reveals that the system possesses two complex conjugate pairs of eigenvalues: $\lambda_1,\ \bar{\lambda}_1 = -2.7506 \pm 3.5643i$ and $\lambda_2,\ \bar{\lambda}_2 = -0.9116 \pm 1.8370i$. From the undamped eigenproblem (2), the system's lowest and highest natural frequencies are, respectively, $\omega_1 = 1.6147$

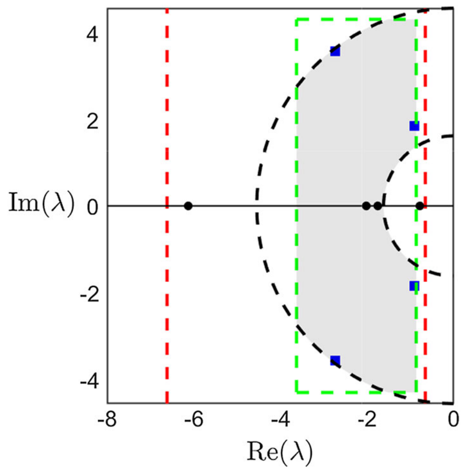

Fig. 1 Location in the complex plane of the real (Filled black circle) and complex (Filled blue square) eigenvalues of the damped linear system in the example of Sect. 5. The annular region containing the complex eigenvalues is delimited by filled black dotted lines, the rectangular region containing the complex eigenvalues is denoted by filled green dotted lines, and the intersection of these two regions is shaded. Each segment filled red dotted lines identifies the lower and upper bounds for the real eigenvalues. Color figure online

and $\omega_4 = 4.5569$. Since $|\lambda_1| = 4.5022$ and $|\lambda_2| = 2.0507$, it can be seen that the complex eigenvalues are indeed bounded, in magnitude, by the annular region in the complex plane defined in Eq. (35):

$$1.6147 \leq |\lambda_1|,\ |\lambda_2| \leq 4.5569.$$

Solution of the eigenvalue problem (18) with $\mathbf{A} = \mathbf{C}$ and $\mathbf{B} = \mathbf{M}$ yields the extreme eigenvalues $\mu_1^{\mathbf{CM}} = -1.7226$ and $\mu_4^{\mathbf{CM}} = -7.2774$, while solving the eigenvalue problem (18) with $\mathbf{A} = \mathbf{K}$ and $\mathbf{B} = \mathbf{C}$ gives $\mu_1^{\mathbf{KC}} = -0.6523$ and $\mu_4^{\mathbf{KC}} = -3.8267$. It can then be verified that the real and imaginary parts of the complex eigenvalues are in fact bounded by the rectangular region in the complex plane formed by Eqs. (37) and (43). Specifically, for the real parts,

$$-3.6387 \leq \mathrm{Re}(\lambda_1),\ \mathrm{Re}(\lambda_2) \leq -0.8613,$$

and for the imaginary parts,

$$0 < |\mathrm{Im}(\lambda_1)|,\ |\mathrm{Im}(\lambda_2)| \leq 4.3005.$$

The four remaining system eigenvalues are real and distinct: $\lambda_3 = -6.1359$, $\lambda_4 = -2.0130$, $\lambda_5 = -1.7483$, and $\lambda_6 = -0.7785$. Because this system is not classically damped, it can be verified that the roots ρ_j of Eq. (44) are not eigenvalues of this system. For example,

$$\mathbf{v}_5^{\mathrm{T}}\mathbf{M}\mathbf{v}_5\lambda^2 + \mathbf{v}_5^{\mathrm{T}}\mathbf{C}\mathbf{v}_5\lambda + \mathbf{v}_5^{\mathrm{T}}\mathbf{K}\mathbf{v}_5 = 0$$

has roots $\lambda_5 = -1.7483$ and $\rho_5 = -2.0500$, the latter of which is not an eigenvalue of the system. Additionally, it can be checked that all real eigenvalues are indeed bounded according to Eq. (52):

$$-6.6252 < \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 < -0.6523.$$

The location of the system eigenvalues in the complex plane and their computed bounds are shown in Fig. 1.

For comparison purposes, it is possible to compute the sharpest classical bounds, which have been reported by Nicholson [7]:

$$-6.8995 < \mathrm{Re}(\lambda_1),\ \mathrm{Re}(\lambda_2),\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 < -0.3779$$

and

$$0 < |\mathrm{Im}(\lambda_1)|,\ |\mathrm{Im}(\lambda_2)| \leq 4.5008.$$

These bounds are less restrictive than the bounds computed by the methodology developed herein.

6 Conclusions

The major results presented in this paper, which are applicable to damped linear systems, are summarized in the following statements.

1. It was proved that the three variants of Rayleigh's quotient (17) for dissipative systems attain a stationary value at the eigenvectors of their respective generalized eigenvalue problems (18) and are also bounded by the corresponding smallest and largest eigenvalues. Contrary to popular beliefs, these results were shown to be true even when the system is non-classically damped.
2. It was demonstrated that the boundedness property of the Rayleigh's quotients (17) is immediately useful for developing simple bounds for the eigenvalues of dissipative systems, some of which appear to be tighter than those previously reported. An annular region in Eq. (35), delimited by the lowest and highest natural frequencies, and a rectangular region, given by Eqs. (37) and (43), containing the complex eigenvalues were determined. Likewise, it was shown that the real eigenvalues are bounded by Eq. (52). Some of these bounds are novel because they use previously unincorporated extreme eigenvalues, while other results exist in the literature but were obtained here by a simpler and more intuitive approach.

Acknowledgements The authors want to acknowledge Matheus Basílio Rodrigues Fernandes, doctorate candidate at the Aeronautics Institute of Technology, for his comments on an earlier draft of this manuscript.

Funding The authors did not receive support from any organization for the submitted work.

Data Availability Statement Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Humar, J.L.: Dynamics of Structures, 2nd edn. Taylor and Francis/Balkema, AK Leiden, The Netherlands (2002)
2. Rayleigh, J.W.S.: The Theory of Sound. Dover, New York (1945). Reprint of the 1894 edition
3. Demmel, J.W.: Applied Numerical Linear Algebra. SIAM, Philadelphia (1997)
4. Parlett, B.N.: The Symmetric Eigenvalue Problem. SIAM, Philadelphia (1998)
5. Phani, A.S., Adhikari, S.: Rayleigh quotient and dissipative systems. ASME J. Appl. Mech. (2008). https://doi.org/10.1115/1.2910898
6. Woodhouse, J.: Linear damping models for structural vibration. J. Sound Vib. **215**(3), 547–569 (1998). https://doi.org/10.1006/jsvi.1998.1709
7. Nicholson, D.W.: Eigenvalue bounds for linear mechanical systems with nonmodal damping. Mech. Res. Commun. **14**(2), 115–122 (1987). https://doi.org/10.1016/0093-6413(87)90028-0
8. Nicholson, D.W.: Stable response of nonclassically damped mechanical systems. ASME Appl. Mech. Rev. **40**(6), 733–740 (1987). https://doi.org/10.1115/1.3149535
9. Hu, B., Eberhard, P.: Response bounds for linear damped systems. J. Appl. Mech. **66**(4), 997–1003 (1999). https://doi.org/10.1115/1.2791810
10. Hu, B., Schiehlen, W.: Eigenvalue, frequency response and variance bounds of linear damped systems. Eur. J. Mech. A. Solids **15**(4), 617–646 (1996)
11. Lancaster, P.: Lambda-Matrices and Vibrating Systems. Dover, New York (2002). Reprint of the 1966 edition
12. Nicholson, D.W., Inman, D.J.: Stable response of damped linear systems. Shock Vib. Dig. **15**(11), 19–25 (1983)
13. Taussky, O.: Positive-definite matrices and their role in the study of the characteristic roots of general matrices. Adv. Math. **2**(2), 175–186 (1968)
14. Ma, F., Caughey, T.K.: Analysis of linear nonconservative vibrations. ASME J. Appl. Mech. **62**(3), 685–691 (1995). https://doi.org/10.1115/1.2896001
15. Tisseur, F., Meerbergen, K.: The quadratic eigenvalue problem. SIAM Rev. **43**(2), 235–286 (2001). https://doi.org/10.1137/S0036144500381988
16. Brandwood, D.H.: A complex gradient operator and its application in adaptive array theory. In: IEE Proceedings H-Microwaves, Optics and Antennas, vol. 130, pp. 11–16 (1983)
17. Rudin, W.: Principles of Mathematical Analysis, 3rd edn. McGraw-Hill Inc, New York, NY (1976)
18. Caughey, T.K., MEJ, O.: Classical normal modes in damped linear dynamic systems. ASME J. Appl. Mech. **32**(3), 583–588 (1965). https://doi.org/10.1115/1.3627262